1 a) $\psi_n = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$ $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$

10

$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \Rightarrow \frac{d^2\psi}{dx^2} = -k^2\psi$ 1 $k \equiv \frac{\sqrt{2mE}}{\hbar}$ 1

Solutions: $\psi(x) = A\sin(kx) + B\cos(kx)$ 1

1 $\psi(x=0) = 0 \Rightarrow B = 0$ 1

1 $\psi(x=a) = 0 \Rightarrow A\sin(ka) = 0$ 1 $\Rightarrow ka = n\pi$ 1 $n=1,2,3$

$k = \frac{n\pi}{a}$ 1

$\frac{n\pi}{a} = \frac{\sqrt{2mE}}{\hbar}$ 1

$$\Rightarrow \boxed{\psi(x) = A\sin\left(\frac{n\pi x}{a}\right)}$$

$$\boxed{E = \frac{n^2\pi^2\hbar^2}{2ma^2}}$$

1 b) $\phi(x) = \begin{cases} A & 0 < x < a \\ 0 & \text{otherwise} \end{cases}$

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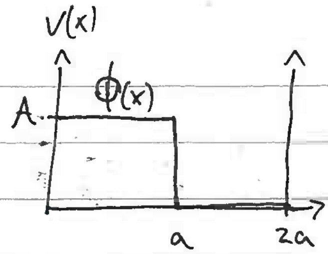

$\Psi(x,t) = ?$

Normalizing ϕ: $\displaystyle\int_{-\infty}^{\infty} |\phi|^2 dx = 1 = |A|^2 \int_0^a dx = |A|^2 a = 1$

$|A|^2 = 1/a \Rightarrow A = 1/\sqrt{a}$ 2

$$\boxed{\Psi(x,t) = \underbrace{\sum_n c_n \psi_n}_{2} \underbrace{e^{-i\epsilon_n t/\hbar}}_{+1}} = 3$$

with $\boxed{\psi_n(x) = \frac{1}{\sqrt{a}} \sin\left(\frac{n\pi x}{2a}\right)}$ 1.5 and $\boxed{\epsilon_n = \frac{n^2 \pi^2 \hbar^2}{2m(2a)^2}}$ 1.5

$$c_n = \int_{-\infty}^{\infty} \psi_n^*(x)\, \phi(x)\, dx \quad 2$$

$$c_n = \frac{1}{\sqrt{a}} \frac{1}{\sqrt{a}} \int_0^a \sin\left(\frac{n\pi x}{2a}\right) dx = \frac{1}{a} \left[-\frac{\cos\left(\frac{n\pi x}{2a}\right)}{n\pi/2a} \Bigg|_0^a \right]$$

$$= \frac{1}{a} \frac{2a}{n\pi} \left[-\cos\left(\frac{n\pi}{2}\right) + (1) \right] \Rightarrow \boxed{c_n = \frac{2}{n\pi}\left(1 - \cos\left(\frac{n\pi}{2}\right)\right)}$$

2

1 c) $E = \dfrac{9\pi^2\hbar^2}{8ma^2} = \dfrac{3^2\pi^2\hbar^2}{2m(2a)^2}$ so $n=3$ 4

10

Probability $|c_3|^2 = \left|\dfrac{2}{3\pi}\left(1-\cos\left(\dfrac{3\pi}{2}\right)\right)\right|^2$ 3

$|c_3|^2 = \dfrac{4}{9\pi^2}$ 3

② a) Simultaneous eigenfunctions of two operators possible if

10

$$[\hat{A}, \hat{B}] = 0 \qquad \text{so is } [L_z, L^2] = 0 \,? \qquad 2$$

$$[L_z, L^2] = [L_z, L_x^2] + [L_z, L_y^2] + [L_z, L_z^2] \qquad 1$$

$$[L_z, L_z^2] = 0 \qquad 1$$

$$[L_z, L_x^2] = L_z L_x L_x - L_x L_x L_z \; (+ L_x L_z L_x - L_x L_z L_x)$$

$$= L_x (L_z L_x - L_x L_z) + (L_z L_x - L_x L_z) L_x$$

$$= L_x [L_z, L_x] + [L_z, L_x] L_x$$

$$= L_x (i\hbar L_y) + (i\hbar L_y) L_x$$

$$[L_z, L_x^2] = i\hbar L_x L_y + i\hbar L_y L_x \qquad 2$$

$$[L_z, L_y^2] = L_z L_y L_y - L_y L_y L_z + L_y L_z L_y - L_y L_z L_y$$

$$= L_y [L_z, L_y] + [L_z, L_y] L_y$$

$$= L_y (-i\hbar L_x) + (-i\hbar L_x) L_y$$

$$[L_z, L_y^2] = -i\hbar L_y L_x - i\hbar L_x L_y \qquad 2$$

So: $[L_z, L^2] = i\hbar L_x L_y + i\hbar L_y L_x - i\hbar L_y L_x - i\hbar L_x L_y + 0$

$$[L_z, L^2] = 0 \;// \qquad 2$$

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(2) b) $l=1$ $\qquad m_l = -1, 0, 1$ 2

A measurement of L_z (or L_x) would result in $+\hbar$, 0, or $-\hbar$.
2.5 2.5

$$L_z Y_l^m = \hbar m_l Y_l^m$$

---

c) $Y(\theta,\phi) = A\sin(\theta)e^{i\phi}$

5/5

$$L_z Y(\theta,\phi) = \hbar m_l Y(\theta,\phi)$$ 2

$$L_z = -i\hbar\, \partial/\partial\phi$$

1

$$-i\hbar \frac{\partial}{\partial\phi}\left(A\sin\theta\, e^{i\phi}\right) = -i\hbar A\sin\theta (i) e^{i\phi} = \hbar A\sin\theta\, e^{i\phi}$$

$L_z Y = \hbar(1) Y$ so $\boxed{m_l = +1}$ 2

8

(2) d) $|\alpha\rangle = |1,1\rangle|1,-1\rangle - |1,-1\rangle|1,1\rangle$

Not normalized. 2

2 $\langle\alpha|\alpha\rangle = \left(\langle 1,1|\langle 1,-1| - \langle 1,-1|\langle 1,1|\right)\left(|1,1\rangle|1,-1\rangle - |1,-1\rangle|1,1\rangle\right)|A|^2$

$$= \left(\langle 1,1|1,1\rangle\langle 1,-1|1,-1\rangle - \langle 1,1|1,-1\rangle\langle 1,-1|1,1\rangle - \langle 1,-1|1,1\rangle\langle 1,1|1,-1\rangle + \langle 1,-1|1,-1\rangle\langle 1,1|1,1\rangle\right)|A|^2$$

$= 1 - 0 - 0 + 1 \Rightarrow \langle\alpha|\alpha\rangle = 2|A|^2$ 2

Normalization constant:

$\boxed{\langle\alpha|\alpha\rangle| = 1} \Rightarrow \boxed{A = \frac{1}{\sqrt{2}}}$ 2

---

3 e) It is an entangled state. 1.5

A measurement of particle 1 changes the state of particle 2. 1.5

3 a) 2

$$H_0 = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} \quad 2$$

$$|\phi_L\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|\phi_R\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

7 b)

$$H_1 = \begin{pmatrix} \varepsilon & T \\ T & \varepsilon \end{pmatrix} \quad 1$$

$$\begin{vmatrix} \varepsilon - E & T \\ T & \varepsilon - E \end{vmatrix} = 0 \Rightarrow (\varepsilon - E)^2 - T^2 = 0$$

1

$$\varepsilon - E = \pm T \Rightarrow E = \varepsilon \mp T$$

Since $T<0$, the ground state energy is $E_g = \varepsilon + T$
and the excited state energy is $E_e = \varepsilon - T$ 2

$$\begin{pmatrix} \varepsilon & T \\ T & \varepsilon \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = (\varepsilon \pm T)\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{matrix} \varepsilon a + Tb = \varepsilon a \pm Ta \\ Ta + \varepsilon b = \varepsilon b \pm Tb \end{matrix}$$

solutions for $a=1$ $b=\pm 1$

So $|\phi_g\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $|\phi_e\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ 1

or normalized: $$\underbrace{|\phi_g\rangle = \frac{1}{\sqrt{2}}\left(|\phi_L\rangle + |\phi_R\rangle\right), \quad |\phi_e\rangle = \frac{1}{\sqrt{2}}\left(|\phi_L\rangle - |\phi_R\rangle\right)}_{2}$$

1 or

3 c) Yes. $\langle \phi_L | \hat{A} | \phi_L \rangle = -a$, $\langle \phi_R | \hat{A} | \phi_R \rangle = +a$ so $\hat{A}|\phi_L\rangle = -a|\phi_L\rangle$, $\hat{A}|\phi_R\rangle = +a|\phi_R\rangle$

2

---

3 d) $\hat{H}_1 \Rightarrow |\phi_g\rangle = \frac{1}{\sqrt{2}}\left(|\phi_L\rangle + |\phi_R\rangle\right)$

Result $-a$ with prob. $|\langle \phi_L | \phi_g \rangle|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = 1/2$ 1.5

Result $+a$ with prob. $|\langle \phi_R | \phi_g \rangle|^2 = 1/2$ 1.5

---

5 e) After the measurement we have state $|\phi_L\rangle$. 2

$$|\phi_g\rangle = \frac{1}{\sqrt{2}}\left(|\phi_L\rangle + |\phi_R\rangle\right)$$
$$|\phi_e\rangle = \frac{1}{\sqrt{2}}\left(|\phi_L\rangle - |\phi_R\rangle\right)$$
$$\Rightarrow |\phi_L\rangle = \frac{1}{\sqrt{2}}\left(|\phi_g\rangle + |\phi_e\rangle\right)$$

3

15 3 f) $\langle \hat{A}(t) \rangle = \langle \phi_L | \hat{U}^\dagger \hat{A} \hat{U} | \phi_L \rangle$ 3

$$\hat{U}|\phi_L\rangle = e^{-i\hat{H}_1 t/\hbar}\left[\frac{1}{\sqrt{2}}\left(|\phi_g\rangle + |\phi_e\rangle\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left(e^{-i\epsilon_g t/\hbar}|\phi_g\rangle + e^{-i\epsilon_e t/\hbar}|\phi_e\rangle\right) \quad 3$$

Define $\omega_g = \frac{\epsilon_g}{\hbar}$ and $\omega_e = \frac{\epsilon_e}{\hbar}$

2 $$\langle \hat{A}(t) \rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\left(e^{i\omega_g t}\langle \phi_g| + e^{i\omega_e t}\langle \phi_e|\right)\hat{A}\left(e^{-i\omega_g t}|\phi_g\rangle + e^{-i\omega_e t}|\phi_e\rangle\right)$$

$$\hat{A}|\phi_g\rangle = \hat{A}\frac{1}{\sqrt{2}}\left(|\phi_L\rangle + |\phi_R\rangle\right) = \frac{1}{\sqrt{2}}\left(-a|\phi_L\rangle + a|\phi_R\rangle\right) = -a|\phi_e\rangle$$

$$\hat{A}|\phi_e\rangle = \hat{A}\frac{1}{\sqrt{2}}\left(|\phi_L\rangle - |\phi_R\rangle\right) = \frac{1}{\sqrt{2}}\left(-a|\phi_L\rangle - a|\phi_R\rangle\right) = -a|\phi_g\rangle$$

2 $$\Rightarrow \langle \hat{A}(t) \rangle = \frac{1}{2}\left(e^{i\omega_g t}\langle \phi_g| + e^{i\omega_e t}\langle \phi_e|\right)\left(-a e^{-i\omega_g t}|\phi_e\rangle - a e^{-i\omega_e t}|\phi_g\rangle\right)$$

$$= \frac{-a}{2}\left(e^{i(\omega_g-\omega_e)t} + e^{i(\omega_e-\omega_g)t}\right) = -a\,\frac{e^{i(\omega_e-\omega_g)t} + e^{-i(\omega_e-\omega_g)t}}{2}$$

$$\boxed{\langle \hat{A}(t) \rangle = -a\cos\left[(\omega_e-\omega_g)t\right]} \quad 2$$

The particle "oscillates" between the left and right wells with frequency $\left(\frac{\epsilon_e-\epsilon_g}{\hbar}\right)$. 3